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HYPERSINGULAR INTEGRALS AND PARABOLIC POTENTIALS

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ABSTRACT. In this paper we characterize the potential spaces associated with the heat equation in terms of singular integrals of mixed homogeneity.

Introduction. Our aim will be to study hypersingular integrals associated with the heat equation. Bagby [1] obtained a characterization for the potential spaces $\mathcal{L}^p_\alpha(\mathbf{R}^{n+1})$ of the heat equation in the range $0 < \alpha < 1$. This characterization is related to that obtained by Strichartz [7] in the case of the usual Riesz potentials. Our characterization is in the range $0 < \alpha < 2$. In view of Lemma 1 this range is enough to prove localization for all $\alpha \geqslant 0$, which is in fact Theorem 2. Our method of proof also generalizes to more general kernels than what has been treated here. The investigation of hypersingular integrals in the case of usual Riesz potentials has been carried out by Stein [5] and Wheeden [8], [9]. We wish to thank the referee for his comments and for his stylistic improvements in the paper.

We wish to recall some notation before stating our main results. All constants appearing depend only on α, p, n.

$\mathbf{R}^{n+1}$ denotes Euclidean space with $n \geqslant 1$.

$\mathbf{R}^{n+1}_+ = \{(x, t) | x \in \mathbf{R}^n, t \geqslant 0\}$.

$\hat{f}(\xi) = \int_{\mathbf{R}^{n+1}} f(x) e^{-i(x\cdot\xi)}\, dx$,

$\Lambda_\varepsilon = \{(x, t) | x \in \mathbf{R}^n_+, |x| \leqslant \varepsilon, 0 \leqslant t \leqslant \varepsilon^2\}$,

$\Omega_\varepsilon = \{(x, t) |\, |x| \leqslant \varepsilon, |t| \leqslant \varepsilon^2\}$.

Ω^c_ε denotes the complement of Ω_ε in $\mathbf{R}^{n+1}$. Λ^c_ε denotes the complement of Λ_ε in $\mathbf{R}^{n+1}_+$. $A - B$ denotes set-theoretic difference.

The Bessel kernel $G_\alpha(x, t)$ is defined as

$$G_\alpha(x, t) = \begin{cases} (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1} t^{(\alpha-n-2)/2} e^{-t-|x|^2/4t}, & t > 0, \\ 0, & t < 0. \end{cases}$$

The Riesz kernel is defined as

$$I_\alpha(x, t) = \begin{cases} (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1} t^{(\alpha-n-2)/2} e^{-|x|^2/4t}, & t > 0, \\ 0, & t < 0. \end{cases}$$

Also, $G_\alpha(g)(x, t) = G_\alpha * g(x, t)$ and $I_\alpha(g)(x, t) = I_\alpha * g(x, t)$, $\mathcal{L}^p_\alpha(\mathbf{R}^{n+1}) = \{f\colon f = G_\alpha(g),\ g \in L^p(\mathbf{R}^{n+1})\}$ and $\|f\|_{p,\alpha} = \|f\|_p + \|g\|_p$. Our main results involve the operator T_ε, where

$$T_\varepsilon f(x, s) = \int\int_{\Lambda^c_\varepsilon} [f(x - y, s - t) - f(x, s)]\, t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt.$$

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The following properties of $G_\alpha(x, t)$ and $I_\alpha(x, t)$ may be found in [3] and [4].

(a) $\hat{G}_\alpha(\xi, t) = (1 + |\xi|^2 + it)^{-\alpha/2}$,

(b) $G_\alpha(x, t) \in L^1(\mathbf{R}^{n+1})$ if $\alpha > 0$,

(c) $\hat{I}_\alpha(\xi, t) = (|\xi|^2 + it)^{-\alpha/2}$ if $0 < \alpha < n + 2$.

There exist finite measures v_1, μ_1, μ_2 such that

(d) $(I + |\xi|^2 + it)^{\alpha/2} = \hat{v}_1 + (|\xi|^2 + it)^{\alpha/2}\hat{\mu}_1$,

(e) $(|\xi|^2 + it)^{\alpha/2} = \hat{\mu}_2(1 + |\xi|^2 + it)^{\alpha/2}$.

Using an argument as in [5, p. 126] we have

LEMMA 1. *$f \in \mathcal{L}^p_\alpha(\mathbf{R}^{n+1})$ if and only if $f \in \mathcal{L}^p_{\alpha-1}(\mathbf{R}^{n+1})$ and $\partial f/\partial x_i \in \mathcal{L}^p_{\alpha-1}(\mathbf{R}^{n+1})$, $i = 1, \ldots, n$, and $\partial f/\partial t \in \mathcal{L}^p_{\alpha-2}(\mathbf{R}^{n+1})$ where $1 < p < \infty$ and $\alpha \geq 2$. Also the norms*

$$\|f\|_{p,\alpha} \quad \textit{and} \quad \|f\|_{p,\alpha-1} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{p,\alpha-1} + \left\| \frac{\partial f}{\partial t} \right\|_{p,\alpha-2}$$

are equivalent.

The following constitute the main results of this paper.

THEOREM 1. *If $f = I_\alpha(g)(x, t)$ where $g \in L^p(\mathbf{R}^{n+1})$, $0 < \alpha < 2$, then:*

(a) *$\|T_\varepsilon f\|_p \leq C_{\alpha,p,n}\|g\|_p$ if $1 < p < \infty$.*

(b) *If $p = 1$ then $m\{x: |T_\varepsilon f| > \lambda\} \leq (C/\lambda)\|g\|_1$.*

(c) *If $g \in L^p$, $1 < p < \infty$, $0 < \alpha < 2$, then $T_\varepsilon f$ is convergent in L^p norm as $\varepsilon \to 0$.*

(d) *If $1 \leq p < \infty$, $0 < \alpha < 2$ and $T_\varepsilon f(x, s)$ is convergent in L^p norm, then $f = I_\alpha(g)$, where $g = \mathrm{Lt}_{\varepsilon \to 0} T_\varepsilon f$, where the limit is in the L^p sense.*

Using the properties of the Riesz and Bessel potentials, we have

COROLLARY 1. *$f \in \mathcal{L}^p_\alpha(\mathbf{R}^{n+1})$, $1 < p < \infty$, $0 < \alpha < 2$, if and only if $T_\varepsilon f(x, s)$ is convergent in L^p norm and $f \in L^p(\mathbf{R}^{n+1})$. Moreover, $\|T_\varepsilon f\|_p \leq C_{\alpha,p,n}\|f\|_{p,\alpha}$.*

PROOF. Since $f \in \mathcal{L}^p_\alpha(\mathbf{R}^{n+1})$, $f = G_\alpha(g)$. But we know that $G_\alpha(g) = I_\alpha(\mu_2 * g)$, where μ_2 is a finite measure. Hence $G_\alpha(g) = I_\alpha(h)$, $h \in L^p(\mathbf{R}^{n+1})$. Using (c) of Theorem 1 we are done. Conversely let $T_\varepsilon f(x, s)$ be convergent in L^p norm. By Theorem 1(d) $f = I_\alpha(g)$. By property (d) we have $f = G_\alpha(v_1 * f + \mu_1 * g)$, hence the result.

We also note that (c) of Theorem 1 follows from (a) by a standard argument using the density of $C_0^\infty(\mathbf{R}^{n+1})$ in $L^p(\mathbf{R}^{n+1})$.

THEOREM 2. *Let $\eta \in C_0^\infty(\mathbf{R}^{n+1})$. If $f \in \mathcal{L}^p_\alpha(\mathbf{R}^{n+1})$, $\alpha \geq 0$, $1 < p < \infty$, then $\eta f \in \mathcal{L}^p_\alpha(\mathbf{R}^{n+1})$; moreover,*

$$\|\eta f\|_{p,\alpha} \leq C_{\alpha,p,n}\|f\|_{p,\alpha}.$$

Our aim in §1 will be to prove part (a) with $p = 2$. This will follow by the observation that if $f = I_\alpha(g)$ then $T_\varepsilon f = K_\varepsilon * g$, where

$$K_\varepsilon(x, s) = \int\int_{\Lambda_\varepsilon^c} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \left[I_\alpha(x - y, s - t) - I_\alpha(x, s) \right] dy\, dt.$$

Hence to obtain L^2 results it is enough to estimate the Fourier transform of $K_\varepsilon(x, s)$. In §2 we will prove a Hörmander condition holds on $K_\varepsilon(x, s)$. This will give part (b), using Lemma 2 of [2]. The idea will be to introduce odd kernels which will assist in the cancellation near the origin. §3 is a proof of part (d) of Theorem 1 and Theorem 2 is proved in §4.

PROPOSITION 1. *Let*

$$\tilde{K}_\varepsilon(x, t) = \begin{cases} t^{-(\alpha+n+2)/2} e^{-|y|^2/ct}, & (x, t) \notin \Lambda_\varepsilon, \\ 0, & (x, t) \in \Lambda_\varepsilon, \end{cases}$$

where $c > 0$, $\alpha > 0$; *then*

$$\int\int |\tilde{K}_\varepsilon(x, t)|\, dx\, dt = C\varepsilon^{-\alpha}.$$

PROOF. Setting $x = \varepsilon x'$, $t = \varepsilon^2 t'$ we readily get

$$\int\int |\tilde{K}_\varepsilon|\, dx\, dt = \left(\int\int_{(x',t')\notin\Lambda_1} t'^{-(\alpha+n+2)/2} e^{-|x'|^2/ct'}\, dx'\, dt'\right)\cdot \varepsilon^{-\alpha}.$$

Now we wish to show that

$$\int_{(x',t')\notin\Lambda_1} t'^{-(\alpha+n+2)/2} e^{-|x'|^2/ct'}\, dx'\, dt' = C.$$

The integral defining C is broken up as follows

$$\int_{t'>1} + \int_{|x'|>1} t'^{-(\alpha+n+2)/2} e^{-|x'|^2/ct'}\, dx'\, dt' = C_1 + C_2.$$

Now

$$C_1 = c\int_1^\infty t'^{-(\alpha+n+2)/2} t'^{n/2}\, dt' = c,$$

and setting $|x'|^2/ct' = u$, in the integral defining C_2 we get

$$\int_{|x'|>1}\int_0^\infty \left(\frac{cu}{|x'|^2}\right)^{(\alpha+n+2)/2} e^{-u} \frac{|x'|^2}{cu^2}\, du$$

$$= c\int_{|x'|>1} |x'|^{-\alpha-n}\, dx' \int_0^\infty u^{(\alpha+n)/2-1} e^{-u}\, du$$

$$= c\Gamma((\alpha+n)/2).$$

We owe the above simplification of our original proof to the referee.

PROPOSITION 2. *Let us consider*

$$(1)\quad A = \left| I_\alpha(x-y-z, s-t-u) - I_\alpha(x-z, s-u) + \sum_{j=1}^n y_j \frac{\partial I_\alpha}{\partial x_j}(x-z, s-u) \right|$$

where $(z, u) \in \Omega_d$, $(y, t) \in \Lambda_{d/2}$.

(a) *If* $s > 16d^2$, $|x| < 4d$, *then*

$$A \leqslant C(|y|^2 + t)s^{(\alpha-n-4)/2} + dts^{(\alpha-n-5)/2} + d^2t^2s^{(\alpha-n-6)/2}).$$

(b) *If* $s > 16d^2$, $|x| \geqslant 4d$, *then*

$$A \leqslant C\big((|y|^2 + t)s^{(\alpha-n-4)/2}e^{-|x|^2/cs} + dts^{(\alpha-n-5)/2}e^{-|x|^2/cs} + d^2t^2s^{(\alpha-n-6)/2}e^{-|x|^2/cs}\big).$$

(c) *If* $|s| < 16d^2$, $|x| \geqslant 4d$, *then*

$$A \leqslant C\big((|y|^2 + t)|x|^{\alpha-n-4} + dt|x|^{\alpha-n-5} + d^2t^2|x|^{\alpha-n-6}\big).$$

PROOF. Applying the mean value theorem to A we have

$$\begin{aligned} A \leqslant\ & |y|^2\left(\sum_{i,j=1}^{n}\left|\frac{\partial^2 I_\alpha}{\partial x_i \partial x_j}(x - z - \xi, s - \tau - u)\right|\right) \\ & + |t|^2(\partial^2 I_\alpha/\partial t^2)(x - z - \xi, s - \tau - u) \\ & + \sum_{i=1}^{n}|y_i|\,|t|\left|\frac{\partial^2 I_\alpha}{\partial x_i \partial t}(x - z - \xi, s - \tau - u)\right| \\ & + |t|\,|(\partial I_\alpha/\partial t)(x - z - \xi, s - \tau - u)| \end{aligned} \tag{2}$$

where $|\xi| \leqslant |y|$, $|\tau| \leqslant t$.

Since $(y, t) \in \Lambda_{d/2}$, $|y| \leqslant d/2$, $t \leqslant d^2/4$, A can be majorized by

$$\begin{aligned} & C\big((|y|^2 + t)(s - \tau - u)^{(\alpha-n-4)/2}e^{-|x-y-\xi|^2/5(s-\tau-u)} \\ & + dt(s - \tau - u)^{(\alpha-n-5)/2}e^{-|x-y-\xi|^2/5(s-\tau-u)} \\ & + d^2t^2(s - \tau - u)^{(\alpha-n-6)/2}e^{-|x-y-\xi|^2/5(s-\tau-u)}\big). \end{aligned} \tag{3}$$

To prove (a) note that as $s > 16d^2$, and since $|\tau| \leqslant t \leqslant d^2/4$, and as $|u| < d^2$, $|(s - \tau - u)| \geqslant Cs$, estimating the exponential by 1, we arrive at the result.

To prove (b) note that $|x - y - \xi| \geqslant C|x|$ and, since $s - \tau - u \geqslant Cs$, we easily get (b) from (3). To prove (c) we observe we can easily have

$$(s - \tau - u)^{(\alpha-n-j)/2}e^{-|x-y-\xi|^2/4(s-\tau-u)} \leqslant C|x - y - \xi|^{\alpha-n-j} \quad \text{for } j = 4, 5, 6.$$

Since $|x| \geqslant 4d$, $|y| \leqslant d^2/4$, $|\xi| \leqslant d$ we easily get (c) from (3).

We note that similar estimates can be made for

$$\left|I_\alpha(x - y, s - t) - I_\alpha(x, s) + \sum_{j=1}^{n} y_j \frac{\partial I_\alpha}{\partial x_j}(x, s)\right|.$$

PROPOSITION 3.

$$\int_0^\varepsilon \int_{\mathbf{R}^n} |I_\alpha(x, t)|\, dx\, dt \leqslant C\varepsilon^\alpha. \tag{4}$$

PROOF. In $\int_0^\varepsilon \int_{\mathbf{R}^n} t^{(\alpha-n-2)/2}e^{-|x|^2/4t}\, dx\, dt$, set $x_i/2\sqrt{t} = x_i'$; an easy integration yields (4).

1. The L^2 estimates. Taking the Fourier transform we obtain

$$\widehat{T_\varepsilon f}(\xi, \tau) = \hat{f}(\xi, \tau)\big[\hat{\tilde{K}}_\varepsilon(\xi, \tau) - \hat{\tilde{K}}_\varepsilon(0, 0)\big], \qquad (\xi, \tau) \neq (0, 0).$$

We now try to estimate the term in brackets.

$$(5)\qquad \hat{\tilde{K}}_\varepsilon(\xi, \tau) - \hat{\tilde{K}}_\varepsilon(0, 0) = \int_{\Lambda_\varepsilon^C} \int (e^{-i(\xi, y)} e^{-it\tau} - 1) t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt$$

where $0 < \alpha < 2$.

Making a change of variable in (5) by setting

$$y_i = \big|\, |\xi|^2 + i\tau \big|^{-1/2} y_i', \qquad t = \big|\, |\xi|^2 + i\tau \big|^{-1} t',$$

(5) can be estimated by

$$(6)\qquad \begin{aligned} &\left|\hat{\tilde{K}}_\varepsilon(\xi, \tau) - \hat{\tilde{K}}_\varepsilon(0, 0)\right| \\ &\quad \leqslant C \cdot \big|\, |\xi|^2 + i\tau \big|^{\alpha/2} \left| \int_{\Lambda_{\varepsilon'}^C} \int (e^{-i(\xi', y')} e^{-it'\tau'} - 1) \right. \\ &\qquad\qquad \left. \times\, t'^{-(\alpha+n+2)/2} e^{-|y'|^2/4t'}\, dy'\, dt' \right| \end{aligned}$$

where $\varepsilon' > 0$.

If $\varepsilon' \geqslant 1$ we simply estimate the integrand by $t'^{-(\alpha+n+2)/2} e^{-|y'^2|/4t'}$; using Proposition 1, (6) is majorized by $C\big|\, |\xi|^2 + i\tau\big|^{\alpha/2}$.

If $\varepsilon' < I$ then we break up (6) as follows:

$$\int_{\Lambda_{\varepsilon'}^C} \int (c^{-i(\xi', y')} e^{-it'\tau'} - 1) t'^{-(\alpha+n+2)/2} e^{-|y'|^2/4t'}\, dy'\, dt'$$

$$= \int\int_{\Lambda_1 - \Lambda_{\varepsilon'}} + \int\int_{\Lambda_1^C} = \mathrm{I} + \mathrm{II}.$$

By the argument used to majorize (6) we can take care of II and, hence, $|\mathrm{II}| < C$. To estimate I we introduce odd kernels whose integrals over $\Lambda_1 - \Lambda_{\varepsilon'}$ are zero. We have

$$(7)\qquad \mathrm{II} = \int\int_{\Lambda_1 - \Lambda_{\varepsilon'}} \left(e^{-i(\xi', y')} e^{-it'\tau'} - 1 - \sum_{i=1}^n y_i' \xi_i' \right) t'^{-(\alpha+n+2)/2} e^{-|y'|^2/4t'}\, dy'\, dt'.$$

Observing that $\xi_i' = \xi_i / |(|\xi|^2 + i\tau)|^{1/2}$, $\tau' = \tau / \big|\, |\xi|^2 + i\tau\big|$, we have $|\xi'| \leqslant 1$, $|\tau'| \leqslant 1$.

Using the mean value theorem we estimate the integrand of (7) by

$$C \cdot \left(|y'|^2 + t' \right) \cdot t'^{-(\alpha+n+2)/2} e^{-|y'|^2/4t'} < C.$$

But

$$C \int\int_{\Lambda_1 - \Lambda_{\varepsilon'}} t'^{(2-\alpha-n-2)/2} e^{-|y'|^2/5t'} < C$$

by Proposition 3.

Hence, collecting all these estimates we have that

$$(8)\qquad \left|\widehat{T_\varepsilon f}(\varepsilon, \tau)\right| \leqslant C |\hat{f}(\xi, \tau)| \, \big|\, |\xi|^2 + i\tau\big|^{\alpha/2}, \qquad (\xi, \tau) \neq (0, 0).$$

Since $f = I_\alpha(g)$, $g \in L^2$,

$$\hat{f}(\xi, \tau) = \left(|\xi|^2 + i\tau \right)^{-\alpha/2} \hat{g}(\xi, \tau).$$

Hence, from (8) we have

$$|\widehat{T_\varepsilon f}(\xi, \tau)| \leqslant C|\hat{g}(\xi, \tau)|.$$

Using the Parseval formula we get Theorem 1 for $p = 2$.

To prove Theorem 1 for other p's we use Lemma 2 of [2]. It will follow from this lemma that

$$(9) \quad m\{(x, t): |T_\varepsilon f(x, t)| > \lambda\} \leqslant (C/\lambda)\|g\|_1, \qquad 0 < \alpha < 2, \quad \text{where } f = I_\alpha(g).$$

In order to use the lemma we are required to show that

$$(10) \quad \int\int |K_\varepsilon(x - z, s - u) - K_\varepsilon(x, s)|\, dx\, ds \leqslant C, \qquad (x, s) \in \Omega_{4d}^C, (z, u) \in \Omega_d.$$

Theorem 1 will follow from (9) and the previously established case $p = 2$, by observing as before that $T_\varepsilon f(x, t) = K_\varepsilon * g(x, t)$, $g \in L^p$. Using the Marcinkiewicz interpolation theorem we easily get Theorem 1 for $1 \leqslant p \leqslant 2$ and the rest follows by duality.

2.

LEMMA 2. *Let* $0 < \alpha < 2$; *then*

$$\int\int_{(x,s) \in \Omega_{4d}^C,\, (z,u) \in \Omega_d} |K_\varepsilon(x - z, s - u) - K_\varepsilon(x, s)|\, dx\, ds < C.$$

PROOF.

$$K_\varepsilon(x, s) = \int\int_{\Lambda_\varepsilon^C} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} [I_\alpha(x - y, s - t) - I_\alpha(x, s)]\, dy\, dt;$$

hence,

$$\begin{aligned} K_\varepsilon(x - z, s - u) &- K_\varepsilon(x, s) \\ &= \int\int_{\Lambda_\varepsilon^C} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \\ &\qquad \times [I_\alpha(x - y - z, s - t - u) - I_\alpha(x - y, s - t) \\ &\qquad\qquad - I_\alpha(x - z, s - u) + I_\alpha(x, s)]\, dy\, dt \qquad (11) \\ &= \int\int_{\Lambda_{d/2} - \Lambda_\varepsilon} + \int\int_{\Lambda_{d/2}^C} = \mathrm{I} + \mathrm{II}. \end{aligned}$$

We shall now consider I. We shall introduce odd kernels in the integrand of I so that the integrals converge if $1 \leqslant \alpha < 2$. It is not necessary to introduce these kernels if $0 < \alpha < 1$. Hence,

$$\mathrm{I}(x, s) = \iint_{\Lambda_{d/2}-\Lambda_\varepsilon} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}$$

$$\times\Bigg[I_\alpha(x - y - z, s - t - u) - I_\alpha(x - y, s - t)$$

$$- I_\alpha(x - z, s - u) + I_\alpha(x, s)$$

$$+ \sum_{j=1}^{n} y_j \frac{\partial I_\alpha}{\partial x_j}(x - z, s - u) - \sum_{j=1}^{n} y_j \frac{\partial I_\alpha}{\partial x_j}(x, s)\Bigg]\, dy\, dt.$$

Let us now consider

$$\iint_{\Omega_{4d}^C} |\mathrm{I}(x, s)|\, dx\, ds = \iint_{\Sigma_1} + \iint_{\Sigma_2} = A_1 + A_2.$$

$$\Sigma_1 = \Omega_{4d}^C \cap \{(x, s) |\, |s| > 16d^2\},$$

$$\Sigma_2 = \Omega_{4d}^C \cap \{(x, s) |\, |s| < 16d^2\}.$$

Let us consider Σ_1. If $s < -16d^2$ then clearly $\mathrm{I}(x, s) = 0$. Hence we consider only $s > 16d^2$. We shall further decompose $A_1 = A_{11} + A_{12}$.

$$A_{11} = \iint_{\tilde{\Sigma}_1} |\mathrm{I}(x, s)|\, dx\, ds, \quad A_{12} = \iint_{\tilde{\Sigma}_2} |\mathrm{I}(x, s)|\, dx\, ds,$$

where

$$\tilde{\Sigma}_1 = \Sigma_1 \cap \{(x, s) |\, |x| \leqslant 4d;\, s > 16d^2\},$$

$$\tilde{\Sigma}_2 = \Sigma_1 \cap \{(x, s) |\, |x| \geqslant 4d;\, s > 16d^2\}.$$

We use Proposition 2(a) to estimate the integrand of A_{11} and Proposition 2(b) to estimate the integrand of A_{12}. Hence,

$$A_{11} \leqslant \iint_{\substack{|x| \leqslant 4d \\ s > 16d^2}} \iint_{\Lambda_{d/2}} |\mathrm{I}(x, s)|\, dx\, ds.$$

As $(y, t) \in \Lambda_{d/2}$, using Proposition 2(a) we have

$$A_{11} \leqslant \iint_{\substack{s > 16d^2 \\ |x| \leqslant 4d}} \iint_{\Lambda_{d/2}} \big[s^{(\alpha-n-4)/2} t^{(2-\alpha-n-2)/2} e^{-|y|^2/ct}$$

$$+ d t^{(2-\alpha-n-2)/2} e^{-|y|^2/ct} s^{(\alpha-n-5)/2}$$

$$+ d^3 s^{(\alpha-n-6)/2} t^{(2-\alpha-n-2)/2} e^{-|y|^2/ct} \big]\, dy\, dt\, dx\, ds.$$

Using Proposition 3 we immediately see that $A_{11} \leqslant C$. Using Propositions 1, 3 and 2(b) we can clearly see that $A_{12} \leqslant C$.

We next consider A_2. To estimate A_2 we note that in this range $|x| \geqslant 4d$; consequently $I(x, s)$ can be estimated by Proposition 2(c).

$$A_2 \leqslant \int\int_{\Sigma_2} \int\int_{\Lambda_{d/2}} t^{(2-\alpha-n-2)/2} e^{-|y|^2/4t} \times \left[|x|^{\alpha-n-4} + d|x|^{\alpha-n-5} + d^3|x|^{\alpha-n-6} \right] dy\, dt\, dx\, ds.$$

Using Proposition 3 and the fact $|x| \geqslant 4d$, $|s| < 16d^2$ we can easily show $A_2 \leqslant C$.

So we are left to make estimates on

$$\int\int_{\Omega_{4d}^C} |\mathrm{II}(x, s)|\, dx\, ds = \int\int_{\Sigma_1} |\mathrm{II}(x, s)|\, dx\, ds + \int\int_{\Sigma_2} |\mathrm{II}(x, s)|\, dx\, ds = B_1 + B_2,$$

and Σ_1 and Σ_2 are as before. We shall consider B_2 first. To estimate B_2 we estimate term by term; we only consider one such term as all are similar in spirit. We consider

$$\int_{\Sigma_2}\int_{\Lambda_{d/2}^C} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} I_\alpha(x - y - z, s - t - u)\, dy\, dt\, dx\, ds \tag{12}$$

where $|z| \leqslant d$, $|u| \leqslant d^2$, $|s| \leqslant 16d^2$, $|x| \geqslant 4d$.

We observe that $s - t - u \geqslant 0$ or else I_α vanishes; consequently we have $t \leqslant s - u$. Now $t \geqslant 0$ so $t \leqslant 17d^2$, hence $0 \leqslant s - t - u \leqslant 34d^2$. We shall use Proposition 3 after a change of variable. Setting $s - t - u = w$ in (12) after extending the range of integration in x to all $\mathbf{R}^n$, (12) is majorized by

$$\int_0^{34d^2} w^{\alpha/2-1}\, dw \int_{\Lambda_{d/2}^C} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt. \tag{13}$$

Using Proposition 1, (13) can be estimated by $Cd^\alpha \cdot d^{-\alpha} = C$. So we are left with crucial estimate for B_1.

We first note that if $s < -16d^2$ then

$$\int\int_{\Omega_{4d}^C \cap \{(x,s) | s < -16d^2\}} |\mathrm{II}(x, s)|\, dx\, ds = 0$$

as the integrand vanishes.

So we only consider

$$B_1 = \int\int_{\Omega_{4d}^C \cap \{(x,s) | s > 16d^2\}} |\mathrm{II}(x, s)|\, dx\, ds.$$

We effect a decomposition of $B_1 \leqslant B_{11} + B_{12}$, where

$$B_{11} = \int\int_{\Omega_{4d}^C \cap \{(x,s) | s > 16d^2\}} |\mathrm{II}_1(x, s)|\, dx\, ds,$$

$$B_{12} = \int\int_{\Omega_{4d}^C \cap \{(x,s) | s > 16d^2\}} |\mathrm{II}_2(x, s)|\, dx\, ds.$$

$$\begin{aligned} \mathrm{II}_1(x, s) = \int\int_{\Lambda_{d/2}^C \cap \{t < s/2\}} & t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \\ & \times \left[I_\alpha(x - y - z, s - t - u) - I_\alpha(x - y, s - t) \right. \\ & \left. - I_\alpha(x - z, s - u) + I_\alpha(x, s) \right] dy\, dt \end{aligned} \tag{14}$$

and

$$\mathrm{II}(x, s) = \mathrm{II}_1(x, s) + \mathrm{II}_2(x, s).$$

We shall first estimate

$$\iint_{\Omega_{4d}^c \cap \{s > 16d^2\}} |\mathrm{II}_1(x, s)|\, dx\, ds.$$

From (14) we easily have that

$$\begin{aligned} |\mathrm{II}_1(x, s)| \leqslant \iint_{\Lambda_{d/2}^c \cap \{t < s/2\}} & t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \\ \times \Bigg[\Bigg| & I_\alpha(x - y - z, s - t - u) - I_\alpha(x - y, s - t) \\ & + \sum_{j=1}^n z_j \frac{\partial I_\alpha}{\partial x_j}(x - y, s - t) \Bigg| \\ & + \Bigg| I_\alpha(x - z, s - u) - I_\alpha(x, s) + \sum_{j=1}^n z_j \frac{\partial I_\alpha}{\partial x_j}(x, s) \Bigg| \\ & + \sum_{j=1}^n |z_j| \left| \frac{\partial I_\alpha}{\partial x_j}(x - y, s - t) - \frac{\partial I_\alpha}{\partial x_j}(x, s) \right| \Bigg]\, dy\, dt. \end{aligned}$$

We shall only make estimates on the first term since the remaining terms are handled similarly.

Let us call this integral II_{11}; the other two will be II_{12} and II_{13}.

$$\begin{aligned} \mathrm{II}_{11} = \iint_{\Lambda_{d/2}^c \cap \{t < s/2\}} & t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \\ \times \Bigg| & I_\alpha(x - y - z, s - t - u) - I_\alpha(x - y, s - t) \\ & + \sum z_j \frac{\partial I_\alpha}{\partial x_j}(x - y, s - t) \Bigg|\, dy\, dt. \end{aligned} \tag{15}$$

We need to show that

$$\iint_{\Omega_{4d}^c \cap \{s > 16d^2\}} |\mathrm{II}_{11}(x, s)|\, dx\, ds \leqslant C.$$

We make a few more decompositions and, in fact, show that

$$\iint_{\Omega_{4d}^c \cap \{s > 16d^2\} \cap \{|x| \leqslant 5d\}} |\mathrm{II}_{11}(x, s)|\, dx\, ds \leqslant C, \tag{16}$$

$$\iint_{\Omega_{4d}^c \cap \{s > 16d^2\} \cap \{|x| > 5d\}} |\mathrm{II}_{11a}(x, s)|\, dx\, ds \leqslant C, \tag{17}$$

$$\iint_{\Omega_{4d}^c \cap \{s > 16d^2\} \cap \{|x| > 5d\}} |\mathrm{II}_{11b}(x, s)|\, dx\, ds \leqslant C, \tag{18}$$

where

$$\mathrm{II}_{11a} = \iint_{\Lambda^C_{d/2} \cap \{t<s/2\} \cap \{|x|<2|y|\} \cap \{|x|>5d\}},$$

$$\mathrm{II}_{11b} = \iint_{\Lambda^C_{d/2} \cap \{t<s/2\} \cap \{|x|>2|y|\} \cap \{|x|>5d\}},$$

where the integrand is the same as in (15).

We shall now estimate (16). We can use Proposition 2(a) interchanging z for y and t for u.

Hence, the integrand in (16) can be estimated as

(19)

$$C\left(d^2 s^{(\alpha-n-4)/2} + d^3 s^{(\alpha-n-5)/2} + d^4 s^{(\alpha-n-6)/2}\right) \iint_{\Lambda^C_{d/2}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt.$$

Integrating (19) and using Proposition 1 (19) can easily be estimated by a constant. To estimate (17) we need to analyze (3). Noting the fact that to apply (3) to the integrand of (15) we simply reverse the roles of y and z and, since $t < s/2$, we easily estimate

$$\mathrm{II}_{11a}(x,s) \leqslant \iint_{\Lambda^C_{d/2} \cap \{t<s/2\} \cap \{|x|<2|y|\}} \times \left[d^2 s^{(\alpha-n-4)/2} + d^3 s^{(\alpha-n-5)/2} + d^4 s^{(\alpha-n-6)/2}\right] \times t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt$$

where we estimated the exponentials occurring in (3) by 1. Hence by interchanging the order of integration:

$$\begin{aligned}\iint_{s>16d^2,\,|x|>5d} |\mathrm{II}_{11a}(x,s)|\, dx\, ds &\leqslant \iint_{s>16d^2} |\mathrm{II}_{11a}(x,s)|\, dx\, ds \\ &\leqslant \iint_{\Lambda^C_{d/2}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \int_{2t}^{\infty} \\ &\quad \times \int_{|x|<2|y|} \left[d^2 s^{(\alpha-n-4)/2} + d^3 s^{(\alpha-n-5)/2} + d^4 s^{(\alpha-n-6)/2}\right] dx\, ds \qquad (20) \\ &\leqslant \int_{\Lambda^C_{d/2}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \\ &\quad \times \left[d^2 t^{(\alpha-n-2)/2} + d^3 t^{(\alpha-n-3)/2} + d^4 t^{(\alpha-n-4)/2}\right] |y|^n\, dy\, dt.\end{aligned}$$

Noting that $|y|^n e^{-|y|^2/4t}/t^{n/2} \leqslant e^{-|y|^2/ct}$, (20) can be majorized by

$$\int_{\Omega^C_{d/2}} e^{-|y|^2/ct} \left(\sum_{j=4}^{6} d^{j-2} t^{-(n+j)/2} \right) dy\, dt. \tag{21}$$

Using Proposition 1, (21) is majorized by a constant.

We now estimate (18). Here $|x| \geqslant 5d$ and $|x| \geqslant 2|y|$. Consequently the integrand of (15) is majorized after an application of (3) by

$$C \sum_{j=4}^{6} d^{j-2} s^{(\alpha-n-j)/2} e^{-|x|^2/cs}.$$

Hence (18) is majorized by

$$(22)\qquad \sum_{j=4}^{6} \int_{s>16d^2} d^{j-2} s^{(\alpha-n-j)/2} e^{-|x|^2/cs} \int_{\Lambda_{d/2}^{\complement}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt.$$

A repeated application of Proposition 1 shows (22) is bounded by a constant. We now consider

$$B_{12} = \iint_{\Lambda_{4d}^{\complement} \cap \{s>16d^2\}} |\mathrm{II}_2(x,s)|\, dx\, ds,$$

where

$$\mathrm{II}_2(x,s) = \iint_{(t>s/2)\cap \Lambda_{d/2}^{\complement}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \times \big[I_\alpha(x-y-z, s-t-u) - I_\alpha(x-y, s-t) - I_\alpha(x-z, s-u) + I_\alpha(x,s) \big]\, dy\, dt.$$

We decompose $\mathrm{II}_2 = \mathrm{II}_{21} + \mathrm{II}_{22}$, where

$$\mathrm{II}_{21} = \iint_{\{t>s/2\}\cap \Lambda_{d/2}^{\complement}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \times \big[I_\alpha(x-y-z, s-t-u) - I_\alpha(x-y, s-t) \big]\, dy\, dt.$$

It is easier to dispose of

$$(23)\qquad \begin{aligned} \iint_{\Omega_{4d}^{\complement} \cap \{s>16d^2\}} |\mathrm{II}_{22}(x,s)|\, dx\, ds &\leqslant \iint_{\Omega_{4d}^{\complement} \cap \{s>16d^2\} \cap \{|x|<5d\}} |\mathrm{II}_{22}(x,s)|\, dx\, ds \\ &\quad + \iint_{\Omega_{4d}^{\complement} \cap \{s>16d^2\} \cap \{|x|>5d\}} |\mathrm{II}_{22}(x,s)|\, dx\, ds \\ &= \iint |\mathrm{II}_{22a}| + \iint |\mathrm{II}_{22b}|. \end{aligned}$$

Applying the mean value theorem to

$$(24)\qquad |I_\alpha(x-z, s-u) - I_\alpha(x,s)| \leqslant c\big[d(s-\tau)^{(\alpha-n-3)/2} e^{-|x-\xi|^2/4(s-\tau)} + d^2(s-\tau)^{(\alpha-\bar{n}-4)/2} e^{-|x-\xi|^2/4(s-\tau)} \big],$$

estimating the exponential by 1 and, since $|\tau| \leqslant |u| \leqslant d^2 < s/16$, we majorize

$$(25)\qquad |\mathrm{II}_{22a}| < \iint_{\Lambda_{d/2}^{\complement}} \big[ds^{(\alpha-n-3)/2} + d^2 s^{(\alpha-n-4)/2} \big] t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt.$$

Using Proposition 1, (24) is majorized by

$$|\mathrm{II}_{22a}| \leqslant C(d^{1-\alpha} s^{(\alpha-n-3)/2} + d^{2-\alpha} s^{(\alpha-n-4)/2});$$

hence,

$$\iint_{\substack{s>16d^2 \\ |x|<5d}} |\mathrm{II}_{22a}|\, dx\, ds \leqslant C.$$

To estimate II_{22b} we choose θ such that $\theta < \alpha$ and $0 < \theta < 1$. Using (24) as $|x| \geqslant 5d$,

$$|\mathrm{II}_{22b}| \leqslant C \int\int_{\Lambda_{d/2}^{C} \cap \{t > s/2\}} \left[d s^{(\alpha-n-3)/2} e^{-|x|^2/cs} + d^2 s^{(\alpha-n-4)/2} e^{-|x|^2/cs} \right] \times t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt.$$

Since $t \geqslant s/2$,

$$|\mathrm{II}_{22b}| \leqslant C \int\int_{\Lambda_{d/2}^{C} \cap \{t > s/2\}} \times \left[\left(d s^{(\theta-n-3)/2} e^{-|x|^2/cs} \right) t^{-(\theta+n+2)/2} e^{-|y|^2/4t} + d^2 s^{(\alpha-n-4)/2} e^{-|x|^2/cs} \right] t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt.$$

Since $\theta < 1$, by a repeated application of Proposition 1, $\int\int |\mathrm{II}_{22b}|\, dx\, ds \leqslant C$.

We now consider II_{21}. We effect a decomposition of $\mathrm{II}_{21} = \mathrm{II}_{21a} + \mathrm{II}_{21b}$, where

$$\mathrm{II}_{21a} = \int\int_{\Lambda_{d/2}^{C} \cap \{t > s/2\} \cap \{|s-t| > 4d^2\}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \times \left[I_\alpha(x-y-z, s-t-u) - I(x-y, s-t) \right] dy\, dt.$$

Note now if $s - t < -4d^2$ then the integrand vanishes. Hence we consider only the case $(s-t) > 4d^2$. We first consider II_{21b}:

$$|\mathrm{II}_{21b}| \leqslant \int\int_{\Lambda_{d/2}^{C} \cap \{t > s/2\} \cap \{|s-t| < 4d^2\}} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \times \left[|I_\alpha(x-y-z, s-t-u) + I_\alpha(x-y, s-t)| \right] dy\, dt. \tag{26}$$

Observing that $0 \leqslant s - t - u \leqslant 5d^2$ we have

$$\int\int_{s > 16d^2} |\mathrm{II}_{21b}|\, dx\, ds \leqslant C \int_0^{5d^2} \int_{\mathbf{R}^n} \int\int_{\Lambda_{d/2}^{C}} t^{-(\alpha+n+2)/2} \times e^{-|y|^2/4t} w^{(\alpha-n-2)/2} e^{-|x|^2/4w}\, dy\, dt\, dx\, dw,$$

where we make the change of variable $w = s - t - u$,

$$x_i' = (x_i - y_i - z_i)/2\sqrt{(s-t-u)}$$

and an analogous one for the second term of (26).

Use of Propositions 1 and 3 yields that

$$\int\int_{s > 16d^2} |\mathrm{II}_{21b}|\, dx\, ds \leqslant C.$$

We now consider II_{21a}:

$$\int\int_{s > 16d^2} |\mathrm{II}_{21a}|\, dx\, ds = \int\int_{\substack{s > 16d^2 \\ |x| < 5d}} |\mathrm{II}_{21a}|\, dx\, ds + \int\int_{\substack{s > 16d^2 \\ |x| > 5d}} |\mathrm{II}_{21a}|\, dx\, ds.$$

The first term is handled like (16). The second term is decomposed as follows:

$$\iint_{\substack{s>16d^2\\|x|\geqslant 5d}} |\mathrm{II}_{21a}|\,dx\,ds$$

$$\leqslant \iint_{\substack{s>16d^2\\|x|\geqslant 5d}} \left| \iint_{\Lambda^{C}_{d/2}\cap\{|x-y|<3d\}\cap\{(s-t)>4d^2\}} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t} \right.$$

(27)

$$\left. \times\left[I_\alpha(x-y-z,s-t-u)-I_\alpha(x-y,s-t)\right]dy\,dt\right| v\,dx\,ds$$

$$+\iint_{\substack{s>16d^2\\|x|\geqslant 5d}} \left| \iint_{\Lambda^{C}_{d/2}\cap\{|x-y|\geqslant 3d\}\cap\{(s-t)>4d^2\}} dy\,dt\right| dx\,ds.$$

Using (24) the first term in (27) can be estimated as follows by setting $x-y$ for x and $s-t$ for s and estimating the exponential by 1.

$$\leqslant C\iint_{\substack{s>16d^2\\|x|\geqslant 5d}} \iint_{\Lambda^{C}_{d/2}\cap(s-t)>4d^2\cap\{|x-y|<3d\}} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t}$$

(28)

$$\times\left[d(s-t)^{(\alpha-n-3)/2}+d^2(s-t)^{(\alpha-n-4)/2}\right]dy\,dt\,dx\,ds.$$

Interchanging the order of integration and changing variables by setting $s-t=w$, $t=t$, (28) is majorized by

$$C\iint_{\Lambda^{C}_{d/2}} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t}\iint_{(w\geqslant 4d^2)\cap\{|x-y|<3d\}}$$

$$\times\left[dw^{(\alpha-n-3)/2}+d^2s^{(\alpha-n-4)/2}\right]dw\,dx\,dy\,dt.$$

Using Proposition 1 we have

$$\leqslant Cd^nd^{-\alpha}\int_{d^2}^{\infty}\left(dw^{(\alpha-n-3)/2}+d^2w^{(\alpha-n-4)/2}\right)dw$$

$$\leqslant Cd^nd^{-\alpha}\left[d^{\alpha-n}\right]\leqslant C.$$

To handle the last term in (27) we note that since $|x-y|\geqslant 3d$ and $|\xi|<d$, and since $s-t>4d^2$, $s-t-\tau\geqslant(s-t)/2$ as $|\tau|<d^2$, the mean value theorem gives

$$|I_\alpha(x-y-z,s-t-u)-I_\alpha(x-y,s-t)|$$

$$< C\left[d(s-t)^{(\alpha-n-3)/2}e^{-|x-y|^2/c(s-t)}+d^2(s-t)^{(\alpha-n-4)/2}e^{-|x-y|^2/c(s-t)}\right].$$

Using the θ previously chosen and since $t>s/2$, $|(s-t)/t|\leqslant 3$, the second term in (27) can be estimated as

$$C\iint_{\substack{s>16d^2\\(s-t)>4d^2}} \iint_{\Lambda^{C}_{d/2}} t^{-(\theta+n+2)/2}e^{-|y|^2/4t}$$

(29)

$$\times\left[d(s-t)^{(\theta-n-3)/2}e^{-|x-y|^2/c(s-t)}\right.$$

$$\left.+d^2(s-t)^{(\alpha-n-4)/2}e^{-|x-y|^2/c(s-t)}\right]dy\,dt\,dx\,ds.$$

Setting $(x_i - y_i)/c\sqrt{(s-t)} = x_i'$, $y_i = y_i$, and $s - t = w$, $t = t$, (29) can be majorized as

$$\leqslant c\int\int_{\Lambda_{d/2}^C} t^{-(\theta+n+2)/2}e^{-|y|^2/4t}\int_{4d^2}^{\infty}\left[dw^{\theta/2-3/2} + d^2w^{\alpha/2-2}\right] dw\, dy\, dt \leqslant cd^{-\theta}d^{\theta} \leqslant c$$

as $0 < \theta < 1$.

The proof of Lemma 1 is now complete.

3. We now assume the convergence in L^p norm of $T_\varepsilon f(x, s)$ as $\varepsilon \to 0$. We shall compute the Fourier transform of the associated convolution kernel as $\varepsilon \to 0$. Let $f \in \mathcal{S}(\mathbf{R}^{n+1})$, the space of rapidly decreasing functions. Then

$$\widehat{T_\varepsilon f}(\xi, \tau) = \hat{f}(\xi, \tau)\left[\hat{\tilde{K}}_\varepsilon(\xi, \tau) - \hat{\tilde{K}}_\varepsilon(0, 0)\right].$$

We will assume that $\xi \neq 0$ to avoid difficulties later on. We will show that as $\varepsilon \to 0$ if $\xi \neq 0$,

$$\hat{\tilde{K}}_\varepsilon(\xi, \tau) - \hat{\tilde{K}}_\varepsilon(0, 0) \to c(\alpha)\left(|\xi|^2 + i\tau\right)^{\alpha/2} \quad \text{if } 0 < \alpha < 2,$$

where $c(\alpha) = (2(4\pi)^{n/2}/\alpha)\Gamma(1 - \alpha/2)$. We also choose the value of $(|\xi|^2 + i\tau)^{\alpha/2}$ such that $-\pi < \arg(|\xi|^2 + i\tau)^{\alpha/2} < \pi$.

We note now that this proves Theorem 1(d), since now $f = I_\alpha(g)$, where $g = \mathrm{Lt}_{\varepsilon\to 0} T_\varepsilon f$, a fact readily verifiable by checking on the Fourier transform side, the limit being taken in the L^p sense. Now

$$\hat{\tilde{K}}_\varepsilon(\xi, \tau) - \hat{\tilde{K}}_\varepsilon(0, 0) = \int\int_{\Lambda_\varepsilon^C} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t}\left[e^{-i(\xi,y)}e^{-it\tau} - 1\right] dy\, dt = \mathrm{I} + \mathrm{II},$$

$$\mathrm{I} = \int\int_{t>\varepsilon^2} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t}\left[e^{-i(\xi,y)}e^{-it\tau} - 1\right] dy\, dt,$$

$$\mathrm{II} = \int\int_{\substack{0<t<\varepsilon^2 \\ |y|>\varepsilon}} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t}\left[e^{-i(\xi,y)}e^{-it\tau} - 1\right] dy\, dt.$$

We show that $|\mathrm{II}| \to 0$ as $\varepsilon \to 0$. To show this we introduce odd kernels again. Consequently,

$$\mathrm{II} = \int\int_{\substack{0<t<\varepsilon^2 \\ |y|>\varepsilon}} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t}\left[e^{-i(\xi,y)}e^{-it\tau} - 1 - i\sum_{i=1}^{n}\xi_i y_i\right] dy\, dt.$$

Estimating II, we have by the mean value theorem,

$$\begin{aligned} |\mathrm{II}| &\leqslant c\int_0^{\varepsilon^2}\int_{|y|>\varepsilon} t^{-(\alpha+n+2)/2}e^{-|y|^2/4t} \\ &\qquad \times \left(t|\tau| + |\xi|^2|y|^2 + t^2|\tau|^2 + |\xi|\,|\tau|\,|y|t\right) dy\, dt, \\ |\mathrm{II}| &\leqslant c\int_0^{2}\int_{|y|>\varepsilon} e^{-|y|^2/ct}\left[|\tau|t^{-(\alpha+n)/2} + |\xi|^2 t^{-(\alpha+n)/2}\right. \\ &\qquad \left. + |\tau|^2 t^{(2-\alpha-n)/2} + |\xi|\,|\tau|t^{(3-\alpha-n-2)/2}\right] dy\, dt. \end{aligned} \tag{30}$$

As $0 < \alpha < 2$, use of Proposition 3 shows that as $\varepsilon \to 0$, (30) tends to zero.

We now compute I:

$$\mathrm{I} = \int_{\varepsilon^2}^{\infty} \int_{\mathbf{R}^n} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \left(e^{-i(\xi,y)} e^{-it\tau} - 1\right) dy\, dt.$$

Doing the y-integration first yields

$$= (4\pi)^{n/2} \int_{\varepsilon^2}^{\infty} t^{-\alpha/2-1} \left(e^{-t(|\xi|^2+i\tau)} - 1\right) dt$$

$$= (4\pi)^{n/2} \left(\int_{\varepsilon^2}^{\infty} t^{-\alpha/2-1} e^{-t(|\xi|^2+i\tau)}\, dt - \frac{2}{\alpha} \varepsilon^{-\alpha} \right).$$

Integrating by parts we have

$$= \frac{2}{\alpha} (4\pi)^{n/2} \left[\left(e^{-\varepsilon^2(|\xi|^2+i\tau)} - 1\right) \varepsilon^{-\alpha} - \left(|\xi|^2 + i\tau\right) \int_{\varepsilon^2}^{\infty} t^{-\alpha/2} e^{-t(|\xi|^2+i\tau)}\, dt \right].$$

Since $0 < \alpha < 2$, as $\varepsilon \to 0$, $\varepsilon^{-\alpha}(e^{-\varepsilon^2(|\xi|^2+i\tau)} - 1) \to 0$. Hence we are left to calculate

$$\operatorname*{Lt}_{\varepsilon \to 0} -\frac{2}{\alpha} (4\pi)^{n/2} \left(|\xi|^2 + i\tau\right) \int_{\varepsilon^2}^{\infty} t^{-\alpha/2} e^{-t(|\xi|^2+i\tau)}\, dt.$$

By a contour integration one can readily show that since $0 < \alpha < 2$ and $\xi \neq 0$,

$$\left(|\xi|^2 + i\tau\right) \int_{\varepsilon^2}^{\infty} t^{-\alpha/2} e^{-t(|\xi|^2+i\tau)}\, dt \to \left(|\xi|^2 + i\tau\right)^{\alpha/2} \Gamma\left(1 - \frac{\alpha}{2}\right).$$

Hence we have proved our claim.

4. We now give a proof of Theorem 2.

Since $\mathfrak{L}_\alpha^p(\mathbf{R}^{n+1}) = L^p(\mathbf{R}^{n+1})$ if $\alpha = 0$ the result is obvious if $\alpha = 0$. By Lemma 1 we see it is enough to prove Theorem 2 for $0 < \alpha < 2$.

We shall need two results, however.

PROPOSITION 4. *If $\beta < \alpha$ then $\mathfrak{L}_\alpha^p(\mathbf{R}^{n+1}) \subseteq \mathfrak{L}_\beta^p(\mathbf{R}^{n+1})$ and $\|f\|_{p,\beta} \leqslant C_{\alpha,p,\beta}\|f\|_{p,\alpha}$.*

PROOF. Obvious from the definition and the fact that $G_\delta \in L^1(\mathbf{R}^{n+1})$ if $\delta > 0$.

LEMMA 3. $\|G_\beta(x - y, s - t) - G_\beta(x, s)\|_1 \leqslant C_{\beta,n}(|y|^2 + |t|)^{\beta/2}$ *if* $0 < \beta < 1$.

PROOF. This can be found in [3].

We note in passing that Lemma 3 can be extended to $2 > \beta \geqslant 1$. To be precise,

$$\left\| G_\beta(x - y, s - t) - G_\beta(x, s) + \sum_{i=1}^{n} y_i \frac{\partial G_\alpha}{\partial x_i}(x, s) \right\|_1 \leqslant C_{\beta,n}\left(|y|^2 + |t|\right)^{\beta/2}.$$

We do not give a proof and it is not needed to give Theorem 2.

COROLLARY 2. *If $f \in \mathfrak{L}_\beta^p(\mathbf{R}^{n+1})$, $0 < \beta < 1$, $1 \leqslant p < \infty$, then*

$$\|f(x - y, s - t) - f(x, s)\|_p \leqslant C_{p,\beta,n}\left(|y|^2 + |t|\right)^{\beta/2} \|f\|_{p,\beta}.$$

This is a direct consequence of Lemma 3.

Let us now proceed to pick β. We choose β such that $\beta + 1 > \alpha$, and $0 < \beta < \alpha$. We can, moreover, find such a β in (0, 1). This is possible since $0 < \alpha < 2$.

Our aim is to show that given $f \in \mathfrak{L}^p_\alpha(\mathbf{R}^{n+1})$ and if

$$T_\varepsilon(\eta f)(x, s) = \iint_{\Lambda_\varepsilon^C} [\eta(x - y, s - t) f(x - y, s - t) - \eta(x, s) f(x, s)] \times t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt,$$

then $T_\varepsilon(\eta f)$ is convergent in L^p norm, $1 < p < \infty$, $0 < \alpha < 2$. We show $T_\varepsilon(\eta f)$ is Cauchy in L^p norm. Let $\varepsilon_1 > \varepsilon_2$.

$$\begin{aligned} T_{\varepsilon_1}(\eta f) - T_{\varepsilon_2}(\eta f) &= \iint_{\Lambda_{\varepsilon_1} - \Lambda_{\varepsilon_2}} \left[\eta(x - y, s - t) - \eta(x, s) + \sum y_i \frac{\partial \eta}{\partial x_i}(x, s) \right] \\ &\qquad \times f(x - y, s - t) t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt \\ &\quad + \eta(x, s) \iint_{\Lambda_{\varepsilon_1} - \Lambda_{\varepsilon_2}} [f(x - y, s - t) - f(x, s)] \\ &\qquad \times t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt \\ &\quad + \sum_{i=1}^{n} \iint_{\Lambda_{\varepsilon_1} - \Lambda_{\varepsilon_2}} y_i \frac{\partial \eta}{\partial x_i}(x, s) [f(x - y, s - t) - f(x, s)] \\ &\qquad \times t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Since $f \in \mathfrak{L}^p_\alpha(\mathbf{R}^{n+1})$ and $|\eta(x, s)| \leqslant M$, from Corollary 1, $\|\mathrm{II}\|_p \to 0$ as $\varepsilon \to 0$.

To estimate III, we use the Minkowski integral inequality, Proposition 4 and Corollary 2 to get

$$\|\mathrm{III}\|_p \leqslant c \iint_{\Lambda_{\varepsilon_1} - \Lambda_{\varepsilon_2}} |y| (|y|^2 + t)^{\beta/2} t^{-(\alpha+n+2)/2} e^{-|y|^2/4t} \|f\|_{p,\beta}\, dy\, dt.$$

Hence

$$\begin{aligned} \|\mathrm{III}\|_p &\leqslant c \iint_{\Lambda_{\varepsilon_1} - \Lambda_{\varepsilon_2}} (|y|^{\beta+1} + t^{\beta/2}|y|) t^{-(\alpha+n+2)/2} e^{-|y|^2/4t}\, dy\, dt \|f\|_{p,\beta} \\ &\leqslant \left(\iint_{\Lambda_{\varepsilon_1} - \Lambda_{\varepsilon_2}} t^{(\beta-\alpha-n-1)/2} e^{-|y|^2/ct} \right) \|f\|_{p,\beta}. \end{aligned}$$

Since $\beta + 1 > \alpha$, using Proposition 3 we observe that this integral goes to zero as $\varepsilon_1, \varepsilon_2 \to 0$.

To estimate I, we use the mean value theorem and it easily follows that $\|\mathrm{I}\|_p \to 0$ as $\varepsilon \to 0$. Using Corollary 1 we conclude our result. To extend the result to higher α we use Lemma 1 and Proposition 4.

It can be seen from the technique of proof of Theorem 2, that for individual 'α' the conditions on $\eta(x, s)$ can be weakened. For example in the range $1 \leqslant \alpha < 2$, if $\eta(x, s)$ is chosen to have compact support and is Lipschitz of order γ ($\gamma > \alpha/2$) in s, and $C^{1+\delta}$ ($1 + \delta > \alpha$) in x, then we still have

$$\|\eta f\|_{p,\alpha} \leqslant C_{\alpha,p,n} \|f\|_{\alpha,p,n}.$$

A more direct proof of Theorem 2, which we outline below, has been communicated to the referee by R. J. Bagby.

(a) Theorem 2 is valid for $0 < \alpha < 1$ (by using the characterization of [**1**]).

(b) Theorem 2 is easily proved valid for $\alpha = 2, 4, 6, 8, \ldots$ by taking partial derivatives.

(c) Consequently by the interpolation results of [**1**] the theorem is valid for every $\alpha \geqslant 0$.

(d) Hence the theorem is valid for $-\infty < \alpha < \infty$ by duality.

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